H_∞ OPTIMIZATION OF FIXED STRUCTURE CONTROLLERS

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ABSTRACT

In this paper, the H_∞ optimization problem of fixed structure linear controllers is considered. Applications of the problem include tuning of SISO (Single-Input Single-Output) PID (Proportional plus Integral plus Derivative) controller gains. The proposed algorithm starts with transformation of the original problems into a static output feedback controller synthesis problem, which does not impose any constraints on the controller structure except for its order. Unlike the full-order H_∞ controller synthesis case, the H_∞ optimization problem of static output feedback controllers cannot be reparameterized as a convex optimization problem. The cone complementarity linearization algorithm is used to overcome the nonconvexity problem due to the constraint on the controller order. The proposed algorithm is applied to the design of a SISO PID controller for head positioning of a magnetic hard disk drive.

1 INTRODUCTION

The LMI (Linear Matrix Inequality)-based H_∞ controller synthesis theory (Gahinet and Apkarian, 1994) guarantees that if the controller is allowed to have the same order as the plant and every system matrix of the controller is freely tunable, then the H_∞ optimization problem can be solved by convex optimization and thus the global minimum can be always found. This paper considers the case where the controller has a fixed structure and only its parameters are tunable. The objective of this paper is to extend the standard LMI-based H_∞ synthesis algorithm for full-order controllers to fixed structure controllers.

Applications of the present optimization problem include tuning of SISO (Single-Input Single-Output) PID (Proportional plus Integral plus Derivative) controller gains. Despite of technical advances in the design and implementation of sophisticated controllers, most of industrial controllers still adopt simple structures such as PID control. Application of the H_∞ optimization theory to tuning of parameters of such fixed structure controllers is, therefore, of interest and of practical importance. It should be noted, however, that H_∞ optimization is not the only tuning methodology for PID controllers. Numerous methods for tuning of PID gains have been reported in the literature over years (see, e.g. Aström and Hagglund, 1995) and any fair comparison among those methods is likely to be inconclusive since their derivations are usually based on different criteria and design philosophy. The H_∞ optimization theory has, however, received a great attention over years in the field of control and its applicability to practical problems has been reported. Especially when the robustness of the closed-loop system is one of the main concerns of the designer, H_∞ optimization is quite a useful tool. Our subsequent discussion focuses on the algorithm to solve the given H_∞ controller optimization problem.

A number of research efforts have been reported about H_∞ optimization of PID gains (Malan et. al., 1994; Grassi and Tsakalis, 1996; Kawabe and Tagami, 1997; Saeki et. al., 1998). Since the standard full-order H_∞ controller synthesis algorithm cannot be applied to this problem, they either adopt nonlinear, nonconvex search algorithms that require heavy computations, or simplify the problem by imposing certain assumptions on the cost function or the controller structure, which makes it difficult to apply the algorithm to more general problems. For problems that can be formulated as standard H_∞ optimization problems ("standard" in the sense that the cost function is given by a single H_∞ norm function), the advantages of the approach proposed in this paper are clear; first, the proposed algorithm is a natural extension of the standard full-order H_∞ controller synthesis algorithm and only requires iteration of convex optimization. Since convex optimization problems can be solved quite efficiently and reliably by using well-developed techniques in semi-definite pro-

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gramming (SDP) (e.g. Vandenberghe and Boyd, 1996), the proposed approach is substantially faster than any approaches based on nonconvex search algorithms. Furthermore, the proposed approach is not restricted to tuning of PID controllers. It can be applied to tuning of any linear controllers that satisfy certain structural conditions, which will be given later.

The proposed algorithm starts with reconstruction of the closed-loop system model by using linear fractional transformations (LFTs) such that all tunable controller parameters are “extracted” as a full constant block. Then, the optimization problem of controller parameters can be seen as an H_∞ synthesis problem of static output feedback control. This transformation allows us to avoid imposing constraints on the controller structure except for its order.

Unlike the full-order controller synthesis case, the H_∞ synthesis problem of static (or more generally, reduced-order) output feedback controllers cannot be reparameterized as a convex optimization problem (i.e. a minimization problem of a convex function under convex set constraints). Therefore, in general, it is quite difficult to find its global solution. The reduced-order H_∞ synthesis problem has, however, received considerable attention in recent years and several local search algorithms have been proposed to solve this problem: e.g. the alternating projection method (Grigoriadis and Skelton, 1995), the min-max algorithm (Geromel et al., 1994), the D-K iteration and the L-R iteration (Iwasaki and Rotea, 1997). Global search algorithms for this problem are discussed in Goh et. al. (1994) and Yamada and Hara (1997), although they are computationally too expensive in practical applications.

This paper employs the cone complementarity linearization algorithm (El Ghaoui et. al., 1997) to solve this problem. Although the cone complementarity linearization algorithm is a local search algorithm and thus does not always guarantee to find the global minimum, in most practical applications it shows excellent search performance, as shown in El Ghaoui et. al. (1997) with extensive numerical examples.

The remainder of this paper is organized as follows. The proposed H_∞ optimization algorithm for parameters in a fixed structure controller is presented in Section 2. The transformation procedure of the original problem into the H_∞ synthesis problem of static output feedback controllers is shown in Section 2.1 and the synthesis algorithm is presented in Section 2.2. In Section 3, the proposed algorithm is applied to the design of a SISO PID controller for head positioning of a magnetic hard disk drive. It shows an example of simple and yet quite useful applications of H_∞ optimization to the design of practical controllers. Simulations results show that the proposed approach can improve controller performance without requiring profound knowledge and experiences for manual loop-shaping.

2 H_∞ OPTIMIZATION ALGORITHM FOR FIXED STRUCTURE CONTROLLERS

2.1 Transformation into a Static Feedback Controller Synthesis Problem

Suppose the closed-loop system is given as shown in Figure 1, where $P(s)$ and $C(s)$ represent a linear plant and controller, respectively. The plant $P(s)$ is partitioned as follows:

$$P(s) = \left[\begin{array}{c|c} P_{11}(s) & P_{12}(s) \\ \hline P_{21}(s) & P_{22}(s) \end{array}\right] \tag{1}$$

where $P_{22}^T(s)$ has the same dimensions as the controller $C(s)$. Suppose the controller dynamics $C(s)$ contains N parameters $k_1,\cdots,k_N$, which are independently tunable. The optimization problem considered in this paper is given as follows:

$$\min_{k_1,\cdots,k_N} ||F_L(P(s),C(s))||_\infty \tag{2}$$

where $F_L(P(s),C(s)) := P_{11}(s)+P_{12}(s)(I-P_{22}(s)C(s))^{-1}P_{21}(s)$, which is the closed-loop transfer function from d to z shown in Figure 1.

In the H_∞ optimization problem of fixed structure controllers, only a part of system matrices of the controller $C(s)$ can be tuned. First, we present the transformation procedure of the original problem (2) into the H_∞ synthesis problem of static output feedback controllers. By using this transformation, the constraints on the controller structure implicitly imposed in the problem (2) can be avoided except for the constraint on the controller order.

To illustrate the procedure, we consider the following SISO PID controller with approximate derivative and integral actions.

$$C(s) = k_p + k_i\frac{1}{s+\tau_i} + k_d\frac{s}{\tau_d s+1} \tag{3}$$

where k_p, k_i and k_d are constant parameters that are independently tunable. τ_i and τ_d are small constants for the approximate integral and derivative actions, respectively, and they are assumed to be fixed. A state space representation of the controller dynamics (3) is given by:

$$\begin{aligned} \begin{bmatrix} \dot{x}_i(t) \\ \dot{x}_d(t) \end{bmatrix} &= \begin{bmatrix} -\tau_i & 0 \\ 0 & -\frac{1}{\tau_d} \end{bmatrix} \begin{bmatrix} x_i(t) \\ x_d(t) \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{1}{\tau_d} \end{bmatrix} e(t) \\ u(t) &= \begin{bmatrix} k_i & k_d(-\frac{1}{\tau_d}) \end{bmatrix} \begin{bmatrix} x_i(t) \\ x_d(t) \end{bmatrix} + \begin{bmatrix} k_p + \frac{k_d}{\tau_d} \end{bmatrix} e(t) \end{aligned} \tag{4}$$

Suppose the plant $P(s)$ is given in the following form:

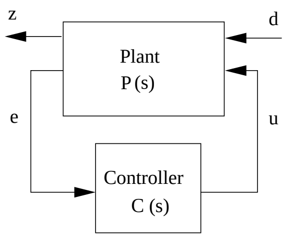


Figure 1. Closed-loop system configuration for the H_∞ optimization problem

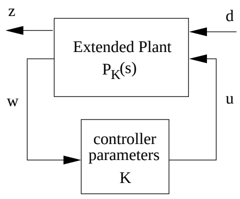


Figure 2. Extraction of controller parameters

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 d(t) + B_2 u(t) \\ z(t) &= C_1 x(t) + D_{11} d(t) + D_{12} u(t) \\ e(t) &= C_2 x(t) + D_{21} d(t) \end{aligned} \tag{5}$$

Then, define the extended plant $P_K(s)$ as follows:

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{x}_i(t) \\ \dot{x}_d(t) \end{bmatrix} &= \begin{bmatrix} A & 0 & 0 \\ C_2 & -\tau_i & 0 \\ \frac{1}{\tau_d} C_2 & 0 & -\frac{1}{\tau_d} \end{bmatrix} \begin{bmatrix} x(t) \\ x_i(t) \\ x_d(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ D_{21} \\ \frac{1}{\tau_d} D_{21} \end{bmatrix} d(t) \\ &\quad + \begin{bmatrix} B_2 \\ 0 \\ 0 \end{bmatrix} u(t) \\ z(t) &= \begin{bmatrix} C_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_i(t) \\ x_d(t) \end{bmatrix} + D_{11} d(t) + D_{12} u(t) \\ w(t) &= \begin{bmatrix} C_2 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{\tau_d} C_2 & 0 & -\frac{1}{\tau_d} \end{bmatrix} \begin{bmatrix} x(t) \\ x_i(t) \\ x_d(t) \end{bmatrix} + \begin{bmatrix} D_{21} \\ 0 \\ \frac{1}{\tau_d} D_{21} \end{bmatrix} d(t) \end{aligned} \tag{6}$$

Notice that $P_K(s)$ contains two controller state variables $x_i(s) = \frac{1}{s+\tau_i} e(s)$ and $x_d(s) = \frac{1}{1+\tau_d s} e(s)$. By using the outputs of $P_K(s)$, the control input $u(t)$ can be given by

$$u(t) = \begin{bmatrix} k_p & k_i & k_d \end{bmatrix} w(t) = \begin{bmatrix} k_p & k_i & k_d \end{bmatrix} \begin{bmatrix} e(t) \\ x_i(t) \\ -\frac{1}{\tau_d} x_d(t) + \frac{1}{\tau_d} e(t) \end{bmatrix} \tag{7}$$

The entire closed-loop system can be seen as shown in Figure 2, where $K := \begin{bmatrix} k_p & k_i & k_d \end{bmatrix}$ is a constant matrix.

This transformation implies that the H_∞ optimization problem of the controller parameters can be seen as the H_∞ synthesis problem of static output feedback controllers for the extended plant model $P_K(s)$, which does not impose any constraints on the controller structure except for its order. The algorithm for the H_∞ (sub-)optimization of static output feedback controllers will be presented in the next section.

The above transformation procedure is not restricted to PID controllers. The optimization algorithm proposed in this paper can be applied to any linear controller structures that can be transformed into the form shown in Fig. 2 by using this transformation. We consider the controller structures that satisfy the following condition.

Lemma 1. *Suppose the the controller has a state space representation (A_c, B_c, C_c, D_c) such that the matrix $\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}$ can be rewritten in the following form:*

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = P_1 + P_2 K P_3 \tag{8}$$

where P_1, P_2 and P_3 are constant (i.e. untunable) matrices with appropriate dimensions and K is a full block of independently tunable controller parameters. Then, the closed-loop system can be transformed into the form shown in Figure 2.

Proof: See (Ibaraki and Tomizuka).

Remarks:

1. The PID controller (3) satisfies the above condition with

$$P_1 = \begin{bmatrix} -\tau_i & 0 & 1 \\ 0 & -\frac{1}{\tau_d} & \frac{1}{\tau_d} \\ 0 & 0 & 0 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad P_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\tau_d} & \frac{1}{\tau_d} \end{bmatrix} \tag{9}$$

and $K = \begin{bmatrix} k_p & k_i & k_d \end{bmatrix}$.

2. If the controller has a state space representation (A_c, B_c, C_c, D_c) such that the full matrix K, the entries of which are all independently tunable, can be obtained by eliminating row(s) and/or column(s) of untunable entries from the matrix $\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}$, then the closed-loop system can be transfered into the form shown in Fig. 2. This condition is included in the condition in Lemma 1.

3. Another example of the controller structure transformable into the form shown in Figure 2 is a SISO controller of the

following transfer function.

$$G_c(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{10}$$

where a_i and b_i $(i = 0, \cdots, n-1)$ are independently tunable parameters. Its controllable canonical form is

$$\dot{x}_c(t) = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} x_c(t) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} e(t)$$

$$u(t) = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix} x_c \tag{11}$$

This representation satisfies the condition in the previous remark. K is given by $K = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix}$.

4. Any controller, the system matrices of which are all freely tunable, can be transformed into the form shown in Fig. 2 with $\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}$, no matter what the order of the controller is (i.e. the reduced-order controller synthesis problem can be always transformed into the static output feedback controller synthesis problem).

2.2 H_∞ Optimization of Static Output Feedback Controller

This section presents the algorithm to solve the H_∞ optimization problem of static output feedback controllers. Suppose that the original H_∞ optimization problem of fixed structure controller parameters (2) can be transformed into the following H_∞ optimization problem of the static output feedback gain matrix K, as shown in the previous section.

$$\min_{K \in \Re^{m_2 \times p_2}} ||F_L(P_K(s), K)||_\infty \tag{12}$$

Rewrite the extended plant model $P_K(s)$ as:

$$P_K : \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right) \tag{13}$$

The plant dimensions are given by $A \in \Re^{n \times n}$, $D_{11} \in \Re^{p_1 \times m_1}$ and $D_{22} \in \Re^{p_2 \times m_2}$. The objective of this section is to present an algorithm to find an H_∞ sub-optimal solution $K \in \Re^{m_2 \times p_2}$ for the problem (12) (i.e. to find $K \in \Re^{m_2 \times p_2}$ such that $||F_L(P_K(s), K)||_\infty < \gamma$ for given $\gamma > 0$).

The standard LMI-based H_∞ controller synthesis algorithm (Gahinet and Apkarian, 1994) is based on the following theorem.

Theorem 1. *There exists a dynamic controller $K(s)$ of order k such that $||F_L(P_K(s), K(s))||_\infty < \gamma$ if and only if there exist two symmetric matrices $X \in \Re^{n \times n}$ and $Y \in \Re^{n \times n}$ such that*

$$\mathcal{N}_1^T \begin{bmatrix} AX + XA^T & XC_1^T & B_1 \\ C_1 X & -\gamma I & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{bmatrix} \mathcal{N}_1 < 0 \tag{14}$$

$$\mathcal{N}_2^T \begin{bmatrix} A^T Y + YA & YB_1 & C_1^T \\ B_1^T Y & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{bmatrix} \mathcal{N}_2 < 0 \tag{15}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \tag{16}$$

$$rank\,(XY - I) \leq k \tag{17}$$

where $\mathcal{N}_1 = diag\{N_{12}, I\}, \mathcal{N}_2 = diag\{N_{21}, I\}$, N_{12} and N_{21} are bases of the null space of $\begin{bmatrix} B_2^T & D_{12}^T \end{bmatrix}$ and $\begin{bmatrix} C_2 & D_{21} \end{bmatrix}$, respectively.

In the full-order controller synthesis case, i.e. $k \geq n$, the condition (17) is trivially satisfied and the global solution for the LMI constraints (14)∼(16) can be computed by convex optimization. For the static controller synthesis problem (12), however, we must search for X and Y that satisfy (14)∼(17) with $k = 0$. The rank condition (17) is not a convex function constraint, which makes the reduced-order H_∞ synthesis problem difficult to solve. Notice that the reduced-order H_∞ controller synthesis problem can be rewritten as a rank minimization problem under LMI constraints (El Ghaoui and Gahinet, 1993):

$$\min_{X,Y} rank\,(XY - I) \quad \text{subject to (14)∼(16)} \tag{18}$$

For the static output feedback controller synthesis, the above rank function can be equivalently replaced by the trace function by using the following lemma.

Lemma 2. *Suppose $X \in \Re^{n \times n}$ and $Y \in \Re^{n \times n}$ are symmetric and satisfy (16). Then, $rank\,(XY - I) = 0$ if and only if $tr\,(XY) = n$.*

Proof: See (Apkarian and Tuan, 1999).

Lemma 2 implies that there exists a constant matrix K^* such that $||F_L(P_K(s), K^*)||_\infty < \gamma$ if and only if

$$\min_{X,Y} tr\,(XY) = n \quad \text{subject to (14)∼(16)} \tag{19}$$

This objective function is still not a convex function of X and Y. We employ the cone complementarity linearization algorithm (El Ghaoui et. al., 1997) to solve this problem. By linearizing the cost function with respect to X and Y, we have:

$$\min_{X_i,Y_i} tr\left(X_{i-1}Y_i + X_iY_{i-1}\right) \qquad \text{subject to (14)}\sim\text{(16)} \qquad (20)$$

With X_{i-1} and Y_{i-1} fixed, X_i and Y_i that minimize the trace function in (20) can be found by a convex optimization. This observation suggests the following iterative algorithm to find X and Y that satisfy (19).

Algorithm (Static output feedback H_∞ controller synthesis)

1. Choose initial $X_0 = X_0^T \in \Re^{n\times n}$ and $Y_0 = Y_0^T \in \Re^{n\times n}$ that satisfy (14)∼(16). If there are none, then the problem is infeasible. Set $i = 1$.
2. Solve the convex optimization problem (20) for X_i and Y_i.
3. Set $i = i+1$ and repeat Step 2 until convergence.

The objective function $t_i := tr\left(X_{i-1}Y_i + X_iY_{i-1}\right)$ is non-increasing at each step, i.e. $t_i \le t_{i-1}$ (El Ghaoui et. al., 1997). Since t_i is lower bounded by $2n$, the algorithm at least converges to one of local minimums.
Although this algorithm is a local search algorithm and thus does not always guarantee to find the global minimum, in most practical applications it shows excellent search performance, as reported by El Ghaoui, et. al. (1997) with extensive numerical examples.
Once the optimal X and Y that satisfy (19) are found, the sub-optimal output feedback gain matrix K^*, which makes the closed-loop H_∞ norm (12) less than γ, can be computed by solving the following convex optimization problem for K^* (Gahinet and Apkarian, 1994).

$$\begin{bmatrix} A^TX_{cl}+X_{cl}A & X_{cl}B_1 & C_1^T \\ B_1^TX_{cl} & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{bmatrix} + \begin{bmatrix} C_2^T \\ D_{21}^T \\ 0 \end{bmatrix} K^{*T} \begin{bmatrix} B_2^TX_{cl} & 0 & D_{12}^T \end{bmatrix} + \begin{bmatrix} X_{cl}B_2 \\ 0 \\ D_{12} \end{bmatrix} K^* \begin{bmatrix} C_2 & D_{21} & 0 \end{bmatrix} < 0 \qquad (21)$$

where $X_{cl} = X^{-1} = Y$. Theorem 1 guarantees the existence of a solution K^* for this problem.

3 APPLICATION EXAMPLE

3.1 Model Description

This section presents an application example to illustrate the H_∞ optimization algorithm proposed in this paper. We consider the design of a SISO PID controller for track-following control of a head positioning servo system of a magnetic hard disk drive (HDD).

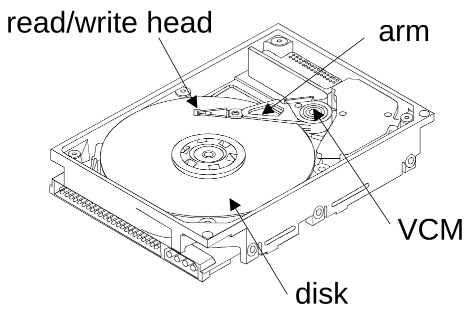


Figure 3. Schematic view of a hard disk drive

Fig. 3 shows the schematic view of an HDD. The head positioning servo system consists of the magnetic read/write head, the arm, and the voice coil motor (VCM) actuator. Fixed servo bursts written on the disks provide information on the deviation of the head from the center of a track.
The head positioning control for an HDD has attracted considerable attention in the literature (e.g. Steinbuch and Norg, 1998; Chew, 1995). Normal operation of an HDD requires quick access to many different tracks. Furthermore, the sensitivity of the head position to external disturbances, such as the vibrations due to the spindle motor rotation, track irregularities, position sensing noise, mechanical vibrations and shocks, is also an important issue for controller design. To satisfy these two different requirements, the controller for the head positioning servo system usually has two modes; the track-seeking mode and the track-following mode. In this paper, we focus on the design of a feedback controller for the track-following mode. Track to track pitches determine the required accuracy of positioning the head; the smaller the pitch, the smaller the error specification. This is a strong technological trend exploited by all HDD manufactures.
The VCM actuator dynamics is modeled as follows.

$$P(s) = K_{vcm}\frac{\omega_1^2}{s^2+2\zeta_1\omega_1 s+\omega_1^2}\cdot\frac{\omega_2^2}{s^2+2\zeta_2\omega_2 s+\omega_2^2}\cdot\frac{-0.5D_1s+1}{0.5D_1s+1} \qquad (22)$$

The first term represents the gain, and the second term represents the double integrator characteristics at high frequencies and flattening characteristics at low frequencies due to the pivot friction. The high resonant mode is also included in the model. The last term $\frac{-0.5D_1s+1}{0.5D_1s+1}$ represents the Pade approximation of computational delay of the controller. The parameters in the model (22) are identified based on measured frequency responses of the experimental setup. The simulated frequency response of the model (22) and the measured frequency response are shown in Fig. 4. The high mechanical resonant mode appears at 3.6 kHz.

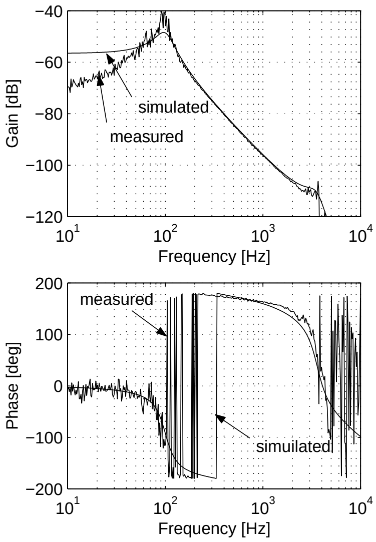


Figure 4. Simulated and measured frequency responses of the VCM actuator

3.2 Controller Design

The objective of this section is to design the PID controller (3) for this plant by using the loop-shaping technique based on the H_∞ optimization algorithm proposed in this paper. The optimal PID controller is obtained as a redesigned version of a second order compensator tuned manually by an expert.

The design requirements are given as follows. 1) The open-loop cross-over frequency should be larger than $f_c = 600$ Hz to secure a sufficient bandwidth of the closed-loop system. 2) The gain and phase margin should be larger than 5 dB and 40 degrees, respectively, for the robustness of the closed-loop system. 3) The peak gain of the closed-loop sensitivity transfer function should be as low as possible to reduce the resonant vibrations.

The following second or third order controller is conventionally used in industrial applications.

$$C_1(s) = K_1 \frac{(s+\omega_{n1})(s+\omega_{n2})}{(s+\omega_{d1})(s+\omega_{d2})} \quad (23)$$

$$C_2(s) = K_2 \frac{(s+\omega_{n1})(s+\omega_{n2})(s+\omega_{n3})}{(s+\omega_{d1})(s+\omega_{d2})(s+\omega_{d3})} \quad (24)$$

where $\omega_{n1} = 170 \times 2\pi$, $\omega_{d1} = 2 \times 10^{-4} \times 2\pi$, $\omega_{n2} = 170 \times 2\pi$, $\omega_{d2} = 7000 \times 2\pi$, $\omega_{n3} = 750 \times 2\pi$. $\omega_{d3} = 1190 \times 2\pi$, $K_1 = 8.334 \times 10^4$ and $K_2 = 10.061 \times 10^4$. These controller parameters are designed by an experienced servo control engineer based on the manual loop-shaping techniques such that the closed-loop system achieves given performance requirements. Note that $C_2(s)$ is designed to reduce the peak gain of the sensitivity transfer function by adding the term $(s+\omega_{n3})/(s+\omega_{d3})$ to introduce additional phase lead around the cross-over frequency. The frequency responses of $C_1(s)$ and $C_2(s)$ are shown in Figure 5 (a).

We design the PID controller (3) by using H_∞ optimization such that it shows "better" performances (with respect to the above design requirements) than the conventional controllers $C_1(s)$ and $C_2(s)$ designed by the manual loop-shaping. Notice that the second order controller $C_1(s)$ is equivalent to the PID controller in the form (3) ($k_p = 1.147 \times 10^4$, $k_i = 6.200 \times 10^6$, $k_d = 5.173$, $\tau_i = 0.0002 \times 2\pi$ and $\tau_d = 1/(7000 \times 2\pi)$), which implies that the optimized PID controller should be at least better than $C_1(s)$. The optimization objective is given as follows:

$$\text{Find } K = (k_p, k_i, k_d) \text{ such that } \left\| \begin{matrix} T(j\omega)W_u(j\omega) \\ S(j\omega)W_p(j\omega) \end{matrix} \right\|_\infty < 1 \quad (25)$$

where $T(j\omega)$ is the closed-loop complementary sensitivity function and $S(j\omega)$ is the closed-loop sensitivity transfer function defined respectively by

$$T(s) = \frac{P(s)C(s)}{1+P(s)C(s)} \quad (26)$$

$$S(s) = \frac{1}{1+P(s)C(s)} \quad (27)$$

where $P(s)$ and $C(s)$ denote the transfer function of the plant and the PID controller, respectively. The constants τ_i and τ_d in (3) are set to $\tau_i = 0.0002 \times 2\pi$ and $\tau_d = 1/(7000 \times 2\pi)$ such that the PID controller has the same poles as $C_1(s)$.

The performance filters $W_u(s)$ and $W_p(s)$ in (25) respectively specify the desired shape of $|T(j\omega)|$ and $|S(j\omega)|$. *We design $W_u(s)$ and $W_p(s)$ based on the actual closed-loop frequency responses of $T(j\omega)$ and $S(j\omega)$ with the conventional controllers $C_2(s)$ (or $C_1(s)$) used in the feedback loop, such that the solution for the problem (25) achieves better performances than the conventional controllers.* This example illustrates quite a simple and yet useful application of the H_∞ optimization to the design of practical controllers.

$W_u(s)$ and $W_p(s)$ are given as follows.

$$W_u(s) = \frac{2.45 \times 10s + 4.62 \times 10^3}{s + 4.05 \times 10^5} \quad (28)$$

$$W_p(s) = \frac{3.25 \times 10^{-1}s^2 + 1.67 \times 10^3 s + 4.09 \times 10^6}{s^2 + 3.90 \times 10^2 s + 3.62 \times 10^4} \quad (29)$$

Their inverse frequency responses are shown in Figure 5 (b) and (c), respectively.

The problem (25) can be transformed into the H_∞ (sub-)optimization problem of static output feedback controllers as shown in Section 2.1. It can be solved by applying the algorithm presented in Section 2.2.

All computations have been carried out by using the SDP solver package *LMI Control Toolbox* (Gahinet et al., 1994) on *MATLAB*. The optimal set of controller parameters $K = \left[\, k_p \ k_i \ k_d \,\right]$ has been obtained after 31 iterations over R_i and S_i. The iteration was terminated when $tr(R_i S_i)$ became less than 10.001. Note that the overall plant $P_K(s)$ is tenth order (the original plant $P(s)$ is fifth order, the controller adds two state variables, and the filters have totally three state variables). The optimal gains $k_p = 4.607 \times 10^3$, $k_i = 5.932 \times 10^6$ and $k_d = 5.864$ achieve the closed-loop H_∞ gain (25) of 1.1130.

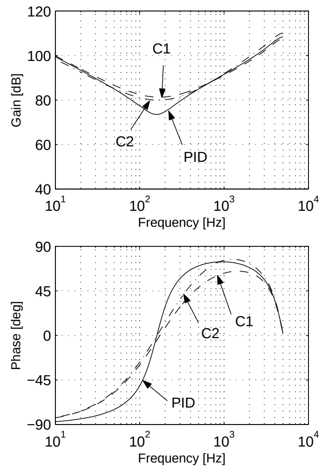
The frequency responses of the designed PID controller $C(s)$, the closed-loop complementary sensitivity transfer function $T(s)$ and the sensitivity transfer function $S(s)$ with the designed PID controller are shown in Figure 5 (a)$\sim$(c). It can be observed from Figure 5(c) that the peak gain of the sensitivity transfer function is reduced compared to the cases where the manually designed controllers $C_1(s)$ and $C_2(s)$ are used. Table 1 shows the order, cross-over frequency, gain margin, phase margin and the peak gain of the sensitivity transfer function for each of three controllers. It shows that the designed PID controller satisfies all of the given performance specifications.

The superiority of the designed PID controller to the conventional controllers, which are finely tuned by an experienced servo engineer, can be explained by its two complex zeros at $s = -3.97 \times 10^2 \pm 9.14 \times 10^2 j$. They introduce additional gain drop at the corresponding frequency and phase lead at higher frequencies to the controller frequency response (see Figure 5 (a)), which leads to desirable drop of the peak gain of the sensitivity transfer function. It is generally difficult to deal with complex zeros/poles by the manual loop-shaping. The "automatic" loop-shaping by the H_∞ optimization has naturally advantages to find a controller that achieves better performance.

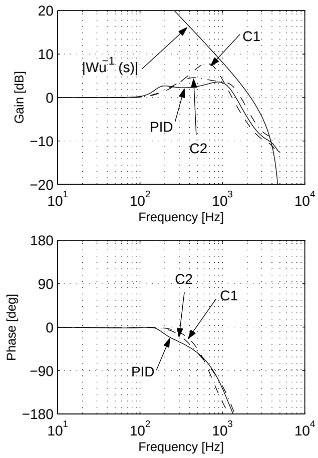
4 CONCLUSION

In this paper, the H_∞ optimization algorithm for parameters of a fixed structure linear controller was presented. The proposed algorithm consists of two steps: 1) "Extraction" of the controller parameters from the closed-loop system as a full constant block by using LFTs and 2) H_∞ synthesis of a static output feedback controller. Unlike the full-order H_∞ controller synthesis case, the H_∞ optimization of static output feedback controllers cannot be solved by convex optimization. The cone complementarity linearization algorithm is used to overcome the nonconvexity problem due to the constraint on the controller order. Although it is a local optimization algorithm, in most cases it shows excellent search performances.

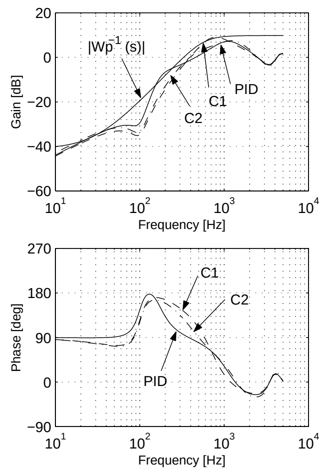
The proposed algorithm was applied to the design of a SISO PID controller for head positioning of a magnetic hard disk drive. The weight functions used in the optimization objective function were chosen such that the optimized controller achieves better performance than the controller conventionally used for this



(a) Controller $C(s)$



(b) Complementary sensitivity transfer function $T(s)$



(c) Sensitivity transfer function $S(s)$

Figure 5. Comparison of closed-loop frequency responses with two conventional controllers and the designed PID controller used in the feedback loop ("$C1$": $C_1(s)$ given in (23), "$C2$": $C_2(s)$ given in (24) and "PID": the PID controller tuned by solving the problem (27))

Table 1. Performance comparison of two conventional controllers (C_1 and C_2) and the designed PID controller (PID)

	C_1	C_2	PID
Order	2	3	2
Cross-over frequency (Hz)	636	622	625
Gain margin (dB)	5.32	5.02	5.53
Phase margin (deg)	23.36	35.82	41.63
Sensitivity function peak (dB)	9.04	7.50	7.20

plant. Simulation results showed effectiveness of the proposed H_∞ optimization algorithm as a fine tuning method to improve controller performance without requiring profound knowledge and experiences in manual loop-shaping.

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